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COMPETITIVE STABILITY UNDER WEAK GROSS SUBSTITUTABILITY:
NONLINEAR PRICE ADJUSTMENT AND ADAPTIVE EXPECTATIONS

BY
KENNETH J. ARROW AND LEONID HURWICZ

TECHNICAL REPORT NO. 78
OCTOBER 12, 1959

PREPARED UNDER CONTRACT Nonr-225(50)
(NR-047-004)
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COMPETITIVE STABILITY UNDER WEAK GROSS SUBSTITUTABILITY:

NONLINEAR PRICE ADJUSTMENT AND ADAPTIVE EXPECTATIONS

by

Kenneth J. Arrow and Leonid Hurwicz¹
Stanford University and the University of Minnesota, *S. F. Paul*

1. Introduction.

In earlier papers by the authors ([4], pp. 545-549), and in collaboration with H. D. Block ([1], pp. 95-104), the global stability of the competitive equilibrium was studied in the case where all commodities are gross substitutes, that is, $\partial F_j / \partial P_k > 0$ for all $j \neq k$, where F_j is the excess demand for commodity j and P_k is the price of commodity k . Two dynamic systems were considered: one a linear system in which the price of each commodity moved proportionately to its excess demand, and the other a more general nonlinear system in which the rate of change of each price was a sign-preserving function of the excess demand (in both cases, with the possible exception of a numéraire). For both systems it was demonstrated that global stability held in the strong sense that for any arbitrary starting point the prices converged to a limit which was necessarily the competitive equilibrium point (which was unique up to a proportionality factor under the assumptions made). For the latter system, the proof of convergence depended upon showing that the "maximum norm,"

¹ Work done with the partial support of a grant from the Rockefeller Foundation to Stanford University for mathematical research in the social sciences.

$$\max_k | (P_k / \bar{P}_k) - 1 | ,$$

where \bar{P} is an equilibrium vector, was necessarily decreasing so long as prices were not at equilibrium.

Subsequently, the stability of nonlinear adjustment processes was studied by Uzawa [15] and McKenzie [12] for the case of weak gross substitutes, i.e., where $\partial F_j / \partial P_k \geq 0$ for $j \neq k$. In this case, the equilibrium may not be unique up to a proportionality factor, but it has been shown by McKenzie ([12], Theorem 1) that the set of equilibria must form a convex set (see also [2], Theorem 2). Both Uzawa and McKenzie made assumptions which implied that there exists at least one equilibrium vector positive in all components. Uzawa assumed that the rate of change of each price (other than the numéraire, if any) was a monotone increasing function of excess demand which vanishes for zero excess demand, a more restrictive dynamic system than that considered in [1]; McKenzie assumed that the rate of change of each price (again other than a possible numéraire) was a function of all prices which, however, had the same sign as the excess demand. Both proved that their respective processes had the property that Uzawa called "quasi-stability" ([16], pp. 3-4): the price movement starting from any initial point is always bounded, and the distance from the moving point to the set of equilibria approaches zero. Equivalently, the second part of the definition can be replaced by the condition that every limit point of the path is an equilibrium. This is a weaker property than convergence to a limit along any path, which we have called (global) stability.

In this paper, we shall consider McKenzie's dynamic system, which is the most general yet proposed, and demonstrate that if there exists at least one strictly positive equilibrium, then the path defined by the dynamic system from any starting point will converge to a limit which, of course, must be an equilibrium. This theorem is stronger than McKenzie's in the sense that the type of stability proved is somewhat stronger. The method of proof is somewhat novel, though related to the methods of [1] and of Uzawa [15].

The results are extended to the case where current excess demand depends upon expected future prices as well as current prices. It is assumed that all commodities, present and future, are weak gross substitutes, and that expectations about future prices are formed from present and past prices according to the principle of adaptive expectations used by Cagan [7], Friedman [9], pp. 143-152, and Nerlove [13] in empirical studies. This hypothesis requires that expected price be changed at a rate proportional to the difference between current actual and current expected price. It has been shown by Nerlove and one of the present authors [5] that local stability can be established for adjustment systems where all commodities are gross substitutes and expectations are adaptive. In this paper, we show that such adjustment systems are globally stable, provided that all equilibria are strictly positive.

We start by proving in Section 2 a general theorem on stability of dynamic systems. This theorem is used in Sections 3 and 5 to prove the stability of the systems without and with expected prices, respectively.

In Section 4, it is shown, by example, that if we do not assume the existence of at least one positive equilibrium, the nonlinear process may lead to unbounded solutions. This possibility does not arise in linear systems, where there is always global stability under conditions of weak gross substitutability, as shown in [2], Theorem 2.

2. A Theorem on Stability.

In this section, we consider a general dynamic system,

$$(1) \quad dP/dt = H(P) ,$$

and state a set of sufficient conditions for the stability of its solutions.

First we state a lemma which is closely related to Lyapunov's "second method" for proving stability and which has been used implicitly in several earlier papers (see especially [3]).

Lemma 1. Constancy of functions on limit paths. Suppose that for any P^0 there is a solution $P(t)$ of equation (1) with $P(0) = P^0$ and that, for fixed t , $P(t)$ is a continuous function of P^0 . Suppose further that $\Phi(P)$ is a continuous function of P , $P(t)$ any solution of (1), and $P^*(t)$ a limit path of $P(t)$, i.e., a solution of (1) with $P^*(0) \equiv P^* =$ a limit point of $P(t)$. Then if $\Phi[P(t)]$ converges to a limit, say Φ^* , $\Phi[P^*(t)] \equiv \Phi^*$, the identity holding in t .

Proof: Use the notation $P(t|P^0)$ to denote the solution with $P(0|P^0) = P^0$. By definition of a limit point, there is a sequence $\{t_n\}$ such that $\lim_{n \rightarrow \infty} P(t_n) = P^*$. From the continuity in the starting point and the uniqueness of the solution,

$$P^*(t) = P(t|P^*) = \lim_{n \rightarrow \infty} P[t|P(t_n)] = \lim_{n \rightarrow \infty} P[t+t_n|P(0)] = \lim_{n \rightarrow \infty} P(t+t_n).$$

From the continuity of Φ ,

$$\Phi[P^*(t)] = \Phi[\lim_{n \rightarrow \infty} P(t+t_n)] = \lim_{n \rightarrow \infty} \Phi[P(t+t_n)] = \Phi^*, \text{ for any } t.$$

Theorem 1. Suppose the system (1) satisfies the following conditions:

- (a) There exists at least one positive equilibrium, that is, a point $\bar{P} > 0$ for which $H(\bar{P}) = 0$;
- (b) for every positive equilibrium \bar{P} and every solution $P(t)$, $\max_j P_j(t) / \bar{P}_j$ is monotone decreasing and $\min_j P_j(t) / \bar{P}_j$ is monotone increasing;
- (c) for any P^0 there is a unique solution $P(t)$ with $P(0) = P^0$; further, for fixed t , $P(t)$ is a continuous function of P^0 ;
- (d) if $P(t)$ is a solution which has at least one component not eventually constant², and $P^*(t)$ a limit path of $P(t)$, then at least one eventually constant component of $P^*(t)$ is not eventually constant in $P(t)$.

Then every solution $P(t)$ of (1) for which $P(0) > 0$ converges to a limit.

Proof: Let the vector P have m components. For any solution $P(t)$, let c be the number of its eventually constant components. The conclusion holds trivially for any solution for which $c = m$. We shall prove it for all solutions by backward induction on c .

Suppose then the theorem is valid for all solutions with $c > c_0$ eventually constant components, and let $P(t)$ be any solution with c_0

² A component, $P_j(t)$, is said to be eventually constant if it is constant for all $t \geq t_0$ for some t_0 .

eventually constant components. Let \bar{P} be any positive equilibrium. From (b) and the assumption that $P(0) > 0$, we have, for all $t \geq 0$,

$$(2) \quad \max_j P_j(0) / \bar{P}_j \geq \max_j P_j(t) / \bar{P}_j \geq P_j(t) / \bar{P}_j \geq \min_j P_j(t) / \bar{P}_j \\ \geq \min_j P_j(0) / \bar{P}_j > 0 .$$

Since $\max_j P_j(t) / \bar{P}_j$ is monotone decreasing and bounded from below, it must approach a limit. We may therefore define

$$(3) \quad \lim_{t \rightarrow \infty} \max_j P_j(t) / \bar{P}_j = \bar{\mu}(\bar{P})$$

A similar remark holds for $\min_j P_j(t) / \bar{P}_j$; since it is monotone increasing from a positive beginning, we can write

$$(4) \quad \lim_{t \rightarrow \infty} \min_j P_j(t) / \bar{P}_j = \underline{\mu}(\bar{P}) > 0 .$$

It follows from (2) that $P(t)$ is bounded and hence has a limit point P^* . Let $P^*(t)$ be the solution with $P^*(0) = P^*$. It follows from (3), (4), and Lemma 1 that

$$(5) \quad \max_j P_j^*(t) / \bar{P}_j \equiv \bar{\mu}(\bar{P}), \min_j P_j^*(t) / \bar{P}_j \equiv \underline{\mu}(\bar{P}) ,$$

the identity holding with respect to t .

For all j such that $P_j(t)$ is eventually constant, it is certainly convergent. By Lemma 1, $P_j^*(t)$ is constant (hence eventually constant). From (d), $P^*(t)$ has at least one eventually constant component that had not been eventually constant in $P(t)$. Therefore, the number of eventually

constant components in $P^*(t)$ is greater than c_0 , and by the induction hypothesis,

$$(6) \quad \lim_{t \rightarrow \infty} P^*(t) = P^{**},$$

for some P^{**} , which must be an equilibrium (see, e.g., [6], Lemma 1, p. 77).

If we replace $P(t)$ by $P^*(t)$ in (2) and use (4) and (5),

$$P_j^*(t) \geq \bar{P}_j \underline{\mu}(\bar{P}) > 0 \text{ for all } t, \text{ so that}$$

(7) P^{**} is a positive equilibrium.

By definition of a limit, $\lim_{t \rightarrow \infty} P_j^*(t)/P_j^{**} = 1$ for all j , so that

$$(8) \quad \lim_{t \rightarrow \infty} \max_j P_j^*(t)/P_j^{**} = 1 = \lim_{t \rightarrow \infty} \min_j P_j^*(t)/P_j^{**}.$$

From (7), (5) holds with $\bar{P} = P^{**}$. By (8), then, $\bar{\mu}(P^{**}) = 1 = \underline{\mu}(P^{**})$.

If in (2) we replace $P(t)$ by $P^*(t)$ and \bar{P} by P^{**} , we see, in view of (5), that $P^*(t) = P^{**}$ for all t . In particular, this holds for $t = 0$, so that $P^* = P^*(0) = P^{**}$, and hence (by (7)),

(9) P^* is a positive equilibrium.

Since P^* was any limit point, the quasi-stability of the system has been shown. However, from the quasi-stability and (b), it will be shown that stability in the stronger sense follows.³ For by definition of a limit point and the positivity of P^* , there must exist a sequence $\{t_n\}$ such

³ Thus, in the theorem of the following section, it would have been possible to infer stability in our sense from the results of Uzawa and McKenzie under their respective assumptions. We adopt the present approach partly as a variant but mainly because it also supplies a technique for handling the case of adaptive expectations.

that $P_j(t_n) / P_j^*$ approaches 1 for each j . Hence,

$$(10) \quad \lim_{t \rightarrow \infty} \max_j P_j(t) / P_j^* = \lim_{t \rightarrow \infty} \min_j P_j(t) / P_j^* = 1.$$

But since $\max_j P_j(t) / P_j^*$ and $\min_j P_j(t) / P_j^*$ both converge, (10) implies that they must both converge to 1. From (2), with $\bar{P} = P^*$, $P_j(t) / P_j^*$ converges to 1 for each j , which demonstrates the conclusion.

3. Stability Under Weak Gross Substitutability Without Expectations.

3.1. We assume there are m commodities, numbered $1, \dots, m$. Let P be the vector of their prices and $F(P)$ the vector of excess demands as a function of P . We make the following assumptions (see, e.g., [2], Sections 1.1 and 1.3):

$$(W) \quad P \cdot F(P) = 0;$$

for each $j = 1, \dots, m$,

$$(H) \quad F_j(P) \text{ is homogeneous of degree } 0,$$

$$(C) \quad F_j(P) \text{ is continuous, and}$$

$$(B) \quad F_j(P) \text{ is bounded from below;}$$

and,

$$(S) \quad \partial F_j / \partial P_k \geq 0 \text{ for all } j \neq k.$$

The dynamic system we assume is that introduced by McKenzie. We assume

$$(D.1) \quad \dot{P}_j = H_j(P) \text{ if } P_j > 0 \text{ or } H_j(P) > 0, \\ = 0, \text{ otherwise,}$$

where, for each j ,

$$(D.2) \quad \text{sgn } H_j(P) = \text{sgn } F_j(P) \text{ for all } P \text{ or } H_j(P) \equiv 0, \text{ the latter}$$

holding for at most one commodity j .

The functions H are assumed continuous.

The dynamic system (D.1-2) is formulated to include both numéraire and non-numéraire systems. The second part of (D.1) insures that any solution is non-negative. Any non-negative ($\neq 0$) initial position is possible, except, of course, that the numéraire price must be positive.

3.2. In this subsection we show that in studying the stability of solutions of (D.1-2) we can confine ourselves to solutions $P(t)$ which are positive everywhere so that, in particular, $P(0) > 0$. It follows that we can disregard the second part of (D.1). We make use of some lemmas proved in [2].

Lemma 2. Let $P(t)$ be any solution of the system (D.1-2), where $F(P)$ satisfies conditions (W), (H), (C), (B), and (S). Then there exists a possibly empty set of indices Z and a time t_0 such that:

- (a) $P_Z(t) \equiv 0$ for $t \geq t_0$;
- (b) $P_{\tilde{Z}}(t) > 0$ for all t ;
- (c) $\dot{P}_{\tilde{Z}} = H_{\tilde{Z}}[0, P_{\tilde{Z}}(t)]$ for $t \geq t_0$;
- (d) $\text{sgn } H_j(0, P_{\tilde{Z}}) = \text{sgn } F_j(0, P_{\tilde{Z}})$ for $j \in \tilde{Z}$, except for the numéraire, if any;
- (e) the function $F_{\tilde{Z}}(0, P_{\tilde{Z}})$ satisfies (H), (C), (B), (W), and (S);
- (f) if \bar{P} is any equilibrium of the excess demand functions $F(P)$, then $\bar{P}_{\tilde{Z}}$ is an equilibrium for the excess demand functions $F_{\tilde{Z}}(0, P_{\tilde{Z}})$ in the space consisting only of the commodities in \tilde{Z} .

Proof: Suppose that on the solution, $P(t)$, commodity j has a zero price at some time $t_1 > 0$. It cannot be that $H_j[P(t_1)] > 0$; for then, by continuity, $H_j[P(t)] > 0$ for $0 \leq t_1 - \varepsilon < t \leq t_1$. By (D.1), $\dot{P}_j(t) > 0$ in this interval. Since $P_j(t_1 - \varepsilon) \geq 0$, $P_j(t_1) > 0$, contrary to assumption. Hence

$$(11) \quad H_j[P(t_1)] \leq 0.$$

Since j cannot be the numéraire (which can never have a zero price because it starts at a positive value and remains constant), it follows from (D.2) that $F_j[P(t_1)] \leq 0$, while $P_j(t_1) = 0$. By Lemma 2 of [2], however, if $F_j(P^1) \leq 0$ for any P^1 for which $P_j^1 = 0$, then $F_j(P) \leq 0$ for all P . Now define

$$(12) \quad Z = \left\{ j: P_j(t) = 0 \text{ for some } t > 0 \right\}.$$

Then we have shown,

$$(13) \quad F_Z(P) \leq 0 \text{ for all } P.$$

It follows from (D.1-2) that $\dot{P}_Z \leq 0$. In particular, if $P_j(t_1) = 0$ for some $j \in Z$, then $P_j(t) = 0$ for all $t \geq t_1$. Since this holds for each $j \in Z$, (a) holds for some t_0 , and (b) follows by definition of Z . By (D.1), (c) follows from (a) and (b), while (d) is a special case of (D.2).

Finally, that (e) and (f) follow from (13) is precisely the assertion of Lemma 3 of [2].

3.3. To prove stability with the aid of Theorem 1, we have to establish or assume conditions (a)-(d). We will assume (a) and (c) hold. We have then to establish (b) and (d). These assertions, which are dynamic in

character, will be shown to follow from the following static lemma. An analogue of this lemma was proved for the case of gross substitutes in the strict sense in [1] (see Lemma 3); a weaker form was established by Uzawa [17], p. 15, Lemma 2. We make use of some steps similar to those in the proof of Theorem 1 of [2].

Lemma 3. If \bar{P} is a positive equilibrium vector, P a positive disequilibrium vector, $\pi(1)$ any permutation of the indices $1, \dots, m$ such that

$$P_{\pi(j)} / \bar{P}_{\pi(j)} \leq (\text{resp. } \geq) P_{\pi(j+1)} / \bar{P}_{\pi(j+1)} ,$$

and J is defined as $\max \{ j : F_{\pi(j)}(P) \neq 0 \}$, then $F_{\pi(j)}(P) < (\text{resp. } >) 0$.

Proof: As in the proof of Theorem 1 of [2], we may assume without loss of generality that $\bar{P}_j = 1$ for all j . Also without loss of generality we may suppose the numbering of the commodities such that $P_j \leq P_{j+1}$ ($j = 1, \dots, m$). Define a sequence of price vectors, P^s , by the relation

$$(14) \quad P_j^s = \min (P_s, P_j) ,$$

so that, in particular, P is an equilibrium vector (with all components equal to P) and $P^m = P$. From (14) and the conventions just made, we see easily that

$$(15) \quad \begin{aligned} P_j^s &= P_j \quad (j \leq s) , \\ &= P_s \quad (j > s) , \end{aligned}$$

$$(16) \quad \begin{aligned} P_j^{s+1} &= P_j \quad (j \leq s) , \\ &= P_{s+1} \quad (j > s) . \end{aligned}$$

Thus a change from P^s to P^{s+1} involves no change in the first \underline{s} prices and an increase (or at least no decrease) in the last $\underline{m - s}$. From (S),

$$(17) \quad F_j(P^{s+1}) \geq F_j(P^s) \quad (j \leq s) .$$

If we write, P_{s+1} being positive by hypothesis,

$$Q^{s+1} = (P_s/P_{s+1}) P^{s+1} ,$$

we see that the last $\underline{m - s}$ components of Q^{s+1} are the same as those of P^s while the first \underline{s} components are smaller or the same. Hence, by (S),

$$F_j(Q^{s+1}) \leq F_j(P^s) \quad (j > s) .$$

But by (H), $F_j(Q^{s+1}) = F_j(P^{s+1})$, so that

$$(18) \quad F_j(P^{s+1}) \leq F_j(P^s) \quad (j > s) .$$

From (18),

$$F_j(P^{s+1}) \leq F_j(P^s) \quad \text{if } j \geq J > s .$$

By induction on \underline{s} , it follows that

$$F_j(P^{s+1}) \leq F_j(P^1) \quad \text{if } j \geq J > s .$$

Since P^1 is a positive equilibrium vector, $F_j(P^1) = 0$. If we set $s = J - 1$, and distinguish the two cases, $j = J$, $j > J$, we have

$$(19) \quad F_J(P^J) \leq 0 ,$$

$$(20) \quad F_j(P^J) \leq 0 \quad \text{for } j > J .$$

From (20),

$$\sum_{j=J+1}^m P_j^J F_j(P^J) \leq 0 .$$

By (W), $\sum_{j=1}^m P_j^J F_j(P^J) = 0$, so that

$$\sum_{j=1}^J P_j^J F_j(P^J) \geq 0 .$$

From (15), $P_j^J = P_j$ for $j \leq J$, so that

$$(21) \quad \sum_{j=1}^J P_j F_j(P^J) \geq 0 .$$

From (17), $F_j(P^{s+1}) \geq F_j(P^s)$ for $j \leq J \leq s$. By induction on s ,

$$F_j(P^J) \leq F_j(P^s) \text{ for } j \leq J \leq s .$$

In particular, let $s = m$, and recall that $P^m = P$.

$$(22) \quad F_j(P^J) \leq F_j(P) \text{ for } j \leq J .$$

Finally, by (W) ,

$$0 = \sum_{j=1}^m P_j F_j(P) = \sum_{j=1}^J P_j F_j(P^J) + \sum_{j=1}^J P_j [F_j(P) - F_j(P^J)] + \sum_{j=J+1}^m P_j F_j(P) .$$

The first summation is non-negative, by (21). Each term of the second summation is non-negative, by (22). Finally, $F_j(P) = 0$ for $j > J$ by hypothesis. Hence, each term in the second summation must be zero, i.e.,

$$F_j(P) = F_j(P^J) \text{ for } j \leq J .$$

In particular, let $j = J$ and recall (19). Since $F_J(P) \neq 0$ by definition, we must have $F_J(P) < 0$.

The other half of the lemma is proved in exactly the same way.

3.4. In this subsection we derive the dynamic implications of Lemma 3; we shall show that the system (D.1-2) has certain properties which in the next subsection will be shown to imply hypotheses (b) and (d) of Theorem 1 and therefore stability.

Lemma 4. Let S be any set of commodities containing the numéraire, if any, and \bar{P} any positive equilibrium. Then: (A) For any given time interval, if $P_j(t)$ is identically constant for all $j \notin S$, then $\max_{j \in S} (\text{resp., } \min_{j \in S}) P_j(t) / \bar{P}_j$ is monotonic decreasing (resp., increasing). (B) If $P_j(t)$ is constant for all $t \geq 0$ for all $j \notin S$, and if $\max_{j \in S} (\text{resp., } \min_{j \in S}) P_j(t) / \bar{P}_j$ is constant for all $t \geq 0$, then there exists a commodity $k \in S$ such that

$$\max_{j \in S} (\text{resp., } \min_{j \in S}) P_j(t) / \bar{P}_j = P_k(t) / \bar{P}_k \text{ for all } t$$

sufficiently large.

Proof:

(A) For convenience, define

$$(23) \quad \bar{V}(P, \bar{P}; S) = \max_{j \in S} P_j / \bar{P}_j ,$$

$$(24) \quad M(t) = \{j: j \in S, P_j(t) / \bar{P}_j = \bar{V}(P(t), \bar{P}; S)\} .$$

For any fixed t' in the interval, let j' be any element of $M(t')$.

Renumber the commodities in increasing order of $P_j(t') / \bar{P}_j$. Since this ratio is the same for all elements of $M(t')$, choose the numbering so that j' is the last element of $M(t')$. Then $P_{j'}(t') / \bar{P}_{j'} > \max_{k \in S} P_k(t') / \bar{P}_k$

for $j > j'$, so that $j \notin S$ for $j > j'$ and therefore $P_j(t)$ is identically constant over the given time interval by hypothesis about elements $\notin S$. By (D.1) and Lemma 2, for $j > j'$, $H_j[P(t)] = 0$ over the interval. Since, for $j > j'$, j cannot be the numéraire (which belongs to S), (D.2) implies that $F_j[P(+)] = 0$ for all points of the time interval and therefore in particular for t' , so that

$$(25) \quad F_j[P(t')] = 0 \quad \text{for } j > j'.$$

But now by Lemma 3, $F_{j'}[P(t')] \leq 0$. Since j' was any element of $M(t')$,

$$(26) \quad F_j[P(t')] \leq 0 \quad \text{for all } j \in M(t').$$

If j is the numéraire, then $H_j \equiv 0$; if not, then from (D.2) and (26), $H_j[P(t')] \leq 0$. Hence, from (D.1),

$$(27) \quad \dot{P}_j \leq 0 \quad \text{for all } j \in M(t') \quad \text{for all } t' \text{ in the interval.}$$

By definition of $M(t)$, $\bar{V}[P(t'), \bar{P}:S] = \bar{V}[P(t'), \bar{P}: M(t')] > P_j(t') / \bar{P}_j$ for all $j \in S - M(t')$. Hence, for t sufficiently close to t' ,

$$(28) \quad \bar{V}[P(t), \bar{P}:S] = \bar{V}[P(t), \bar{P}: M(t')].$$

By definition of $M(t)$,

$$(29) \quad P_j(t') / \bar{P}_j = \bar{V}[P(t'), \bar{P}: M(t')] \quad \text{for all } j \in M(t').$$

We will examine the right- and left-hand derivatives of $\bar{V}[P(t), \bar{P}:S]$ at $t = t'$. From (28),

$$\begin{aligned}
 (30) \quad & \bar{V}[P(t' + h), \bar{P}:S] - \bar{V}[P(t'), \bar{P}:S] \\
 &= \bar{V}[P(t' + h), \bar{P}:M(t')] - \bar{V}[P(t'), \bar{P}:M(t')] \\
 &= \max_{j \in M(t')} P_j(t' + h) / \bar{P}_j - \max_{j \in M(t')} P_j(t') / \bar{P}_j \\
 &= \max_{j \in M(t')} [P_j(t' + h) - P_j(t')] / \bar{P}_j,
 \end{aligned}$$

from (29). For $h > 0$, then,

$$\begin{aligned}
 (31) \quad & \{ \bar{V}[P(t' + h), \bar{P}:S] - \bar{V}[P(t'), \bar{P}:S] \} / h \\
 &= \max_{j \in M(t')} [P_j(t' + h) - P_j(t')] / (h \bar{P}_j).
 \end{aligned}$$

If we let h approach zero from the right, the right-hand derivative is, then,

$$\max_{j \in M(t')} \dot{P}_j / \bar{P}_j \leq 0,$$

by (27). Similarly, the left-hand derivative is

$$\min_{j \in M(t')} \dot{P}_j / \bar{P}_j \leq 0.$$

Hence, both the right- and left-hand derivatives of $\bar{V}[P(t), \bar{P}:S]$ exist and are non-positive, so that $\bar{V}[P(t), \bar{P}:S]$ is monotone decreasing, while a similar argument shows that

$$\min_{j \in S} P_j(t) / \bar{P}_j$$

is monotone increasing; this is assertion (A).

(B) By hypothesis,

$$(32) \quad \bar{V}[P(t), \bar{P}:S] \equiv \bar{\mu},$$

say. If the numéraire j_0 belongs to $M(t)$ for some value of t , then $P_{j_0} / \bar{P}_{j_0} = \bar{\mu}$ and so the numéraire belongs to $M(t)$ for all t , so that (B) holds. Let us now assume that the numéraire belongs to $M(t)$ for no value of t . For each t , $M(t)$ is non-null and so contains a positive integral number of elements. Choose t_0 so that $M(t_0)$ has fewest elements. For simplicity, let

$$(33) \quad M = M(t_0), \quad N = S - M(t_0).$$

By construction, N contains the numéraire. By definition,

$$(34) \quad \bar{V}[P(t), \bar{P}:S] = \max \{ \bar{V}[P(t), \bar{P}:M], \bar{V}[P(t), \bar{P}:N] \},$$

$$(35) \quad \bar{V}[P(t_0), \bar{P}:M] > \bar{V}[P(t_0), \bar{P}:N].$$

Consider any t for which $\bar{V}[P(t), \bar{P}:M] > \bar{V}[P(t), \bar{P}:N]$. From (34), $M(t)$ must be disjoint from N and therefore a subset of M . But $M(t)$ cannot contain fewer elements than $M(t_0)$, so that

$$(36) \quad \text{if } \bar{V}[P(t), \bar{P}:M] > \bar{V}[P(t), \bar{P}:N], \text{ then } M(t) = M.$$

Suppose $\bar{V}[P(t), \bar{P}:N] = \bar{V}[P(t), \bar{P}:S]$ for some $t > t_0$. Let t_1 be the earliest such t . In view of (34) and (35),

$$(37) \quad \bar{V}[P(t), \bar{P}:M] > \bar{V}[P(t), \bar{P}:N] \quad \text{for } t_0 < t < t_1.$$

Hence, by (36), $M(t) = M$ in this interval. From (32), $P_j(t)$ must be identically constant for all $j \in M$ in this interval. If $j \in N$, then $j \in M$ or $j \notin S$; in either case, $P_j(t)$ is constant in the interval. Since N contains the numéraire, we can apply part (A) of this lemma to the set N ; $\bar{V}[P(t), \bar{P}:N]$ is monotone decreasing in (t_0, t_1) , so that

$$\bar{V}[P(t_1), \bar{P}:N] \leq \bar{V}[P(t_0), \bar{P}:N] < \bar{V}[P(t_0), \bar{P}:M] = \bar{\mu} = \bar{V}[P(t_1), \bar{P}:S] ,$$

by (35), (34), and (32), contradicting the definition of t_1 . Hence,

$$\bar{V}[P(t), \bar{P}:S] > \bar{V}[P(t), \bar{P}:N] \text{ for all } t > t_0 ;$$

from (34) and (36), $M(t) = M(t_0)$ for all $t > t_0$. Therefore (B) holds for any $k \in M(t_0)$.

Again the argument for the minimum is entirely parallel.

3.5. The proof of stability is now an easy consequence of Theorem 1 and Lemma 4.

Theorem 2. If $F(P)$ satisfies (W), (H), (C), (B), and (S), and if it possesses a positive equilibrium, then the dynamic system defined by

$$(D.1) \quad \dot{P}_j = H_j(P) \text{ if } P_j > 0 \text{ or } H_j(P) > 0 , \\ = 0 \text{ otherwise,}$$

where

$$(D.2) \quad \text{sgn } H_j(P) = \text{sgn } F_j(P) \text{ for all } P \text{ or } H_j(P) \equiv 0 , \text{ the latter} \\ \text{holding for at most one } j ,$$

is globally stable in the sense that every solution converges, provided the solutions are unique and continuous in the starting point.

Proof: By Lemma 2, we can assume that $\dot{P} = H(P)$ and that $P(0) > 0$.

Conditions (a) and (c) of Theorem 1 have been assumed here; it remains to demonstrate (b) and (d).

If we let S be the set of all commodities $1, \dots, m$, then assertion A of Lemma 4 is hypothesis (b) of Theorem 1.

To verify (d) of Theorem 1, let C be the set of components of $P(t)$ which are eventually constant other than the numéraire, if any. Then for t sufficiently large, \tilde{C} satisfies the conditions for the set S in Lemma 4 (assertion A), so that

$$\max_{j \in \tilde{C}} P_j(t) / \bar{P}_j \text{ is monotone decreasing,}$$

and

$$\min_{j \in \tilde{C}} P_j(t) / \bar{P}_j \text{ is monotone increasing.}$$

Since $\max_{j \in \tilde{C}} P_j(t) / \bar{P}_j \geq \min_{j \in \tilde{C}} P_j(t) / \bar{P}_j \geq 0$, both functions are bounded and hence convergent.

Let $P^*(t)$ be a limit path of $P(t)$. Then, by Lemma 1,

$$(38) \quad \max_{j \in C} P_j^*(t) / \bar{P}_j \equiv \bar{\mu}, \quad \min_{j \in C} P_j^*(t) / \bar{P}_j \equiv \underline{\mu},$$

for suitably chosen constants $\bar{\mu}, \underline{\mu}$.

Also, by definition of the set C , $P_j(t)$ is convergent for $j \in C$. Again by Lemma 1,

$$(39) \quad P_j^*(t) \text{ is identically constant for } j \in C.$$

If $\bar{\mu} = \underline{\mu}$, then $P_j^*(t)$ would be constant for all j in \tilde{C} , from (38). In conjunction with (39), all components of $P^*(t)$ would be constant, so that (d) would certainly be satisfied.

Otherwise, $\bar{\mu} > \underline{\mu}$. From (38), (39), and Lemma 4, assertion (B), there exist commodities j', j'' such that

$$(40) \quad P_{j'}^*(t) / \bar{P}_{j'} \equiv \bar{\mu}, \quad P_{j''}^*(t) / \bar{P}_{j''} \equiv \underline{\mu}, \quad \text{for } t \text{ sufficiently large.}$$

Both j' and j'' are eventually constant. They cannot be the same, since $\bar{\mu} > \underline{\mu}$, and therefore they cannot both be the numéraire. Hence at least one was not eventually constant along $P(t)$. Hence (d) follows, completing the proof of Theorem 2.

4. An Example of Instability in the Absence of a Positive Equilibrium.

When there is gross substitutability in the strict sense, there must be a positive equilibrium (see [1], p. 88, corollary to Lemma 1). However, (S) in its present form is not sufficient to guarantee this, so the assumption of a positive equilibrium is an additional one. Further, it will now be shown by means of an example, that Theorem 2, at least in its present form, would not remain valid if this assumption were dropped.

Suppose there are two commodities, with excess demand functions,

$$(41) \quad F_1(P) = P_2^2/P_1^2, \quad F_2(P) = -P_2/P_1.$$

These excess demand functions satisfy the conditions (W), (H), (C), and (S). They do not satisfy (B) as they stand; however, we can regard (41) as valid only for $P_2/P_1 \leq 1$, and define the excess demand functions for $P_2/P_1 \geq 1$, by

$$F_1 = 2(P_2/P_1) - 1, \quad F_2 = (P_1/P_2) - 2.$$

The modified functions satisfy all conditions. In the example, we shall only need the definition of the functions in the region, $P_2/P_1 \leq 1$.

Note that the only equilibrium points are those for which $P_2 = 0$, $P_1 > 0$. Hence, there is no positive equilibrium.

We now define the adjustment process. Let

$$(42) \quad \Phi(u) = 2u e^{-1/u}.$$

This function is zero when $u = 0$ and positive for all positive values of u , approaching infinity as u approaches infinity. By differentiation, it is easy to see that Φ is strictly increasing. Hence, it has a well-defined inverse $\psi(v)$ defined for all non-negative v with

$$(43) \quad \text{sgn } \psi(v) = \text{sgn } v.$$

We now let

$$(44) \quad H_1(P) = \psi(P_2/P_1), \quad H_2(P) = -P_2/P_1 = F_2(P).$$

From (41), $\text{sgn } F_1 = \text{sgn } (P_2/P_1)$; hence from (43) and (44), $\text{sgn } F_1 = \text{sgn } H_1$, while the condition, $\text{sgn } F_2 = \text{sgn } H_2$, is trivially satisfied. In fact, since $H_1 = \psi(\sqrt{F_1})$, H_1 and H_2 are actually monotonic increasing as well as sign-preserving functions of F_1 and F_2 , respectively.

The dynamic system, $\dot{P} = H(P)$, can be written, in view of the definitions, as

$$(45) \quad \dot{\Phi}(P_1) = P_2/P_1, \quad \dot{P}_2 = -P_2/P_1.$$

It is easily verified that the pair of functions

$$P_1(t) = \sqrt{t+1}, \quad P_2(t) = e^{-2\sqrt{t+1}},$$

constitute a solution of (45). Along this path, $P_2(t)/P_1(t) \leq e^{-2} < 1$, so that confining ourselves to (41) involved no loss of generality. But then $P_1(t)$ approaches infinity and so the solution is not convergent in the usual sense.

Nevertheless it must be noted that there is a kind of convergence, for $P_2(t)$ approaches zero, so that the solution does approach the set

of equilibria (which includes all points with $P_2 = 0$). Put slightly differently, the relative prices converge, though the absolute prices are in part unbounded.

5. Stability Under Expectations.

5.1. We now suppose that the demand for commodities depends on both current prices and future expected prices. Let current prices be denoted by P_j ($j = 1, \dots, m$), while expected future prices are denoted by P_j ($j = m+1, \dots, 2m$), where P_{j+m} is the expected future price of commodity j . The vector P will have $2m$ components; we will also write P^1 for the vector of current prices, P_1, \dots, P_m and P^2 for the vector of future prices.

Given the vector P , each individual, and therefore the market, determines a $2m$ -vector of excess demands, $F(P)$. The first m components, $F^1(P) = F^1(P^1, P^2)$, are excess demands for current goods. The last m components, $F^2(P)$, are planned excess demands for the future. In the absence of futures markets, these planned excess demands can have no influence on prices.⁴ We will then suppose that the dynamic relations (D.1-2) of Section 3 apply only to current prices:

⁴ That current and planned excess demands depend on current and expected prices is, of course, a standard doctrine; see, e.g., Hicks [10] or Lange [11]. These works did not, however, have an explicit formulation of stability as related to price adjustment. Patinkin [14], Chapters IV, X, and their Appendices, and Enthoven [8] have formulated dynamic models in which current excess demands influenced current prices but planned excess demands had no relevance to price movements. As here, it becomes necessary to introduce expectational assumptions; Patinkin uses static assumptions, Enthoven an assumption of extrapolation from current values on the basis of the current rate of change.

$$(DE.1) \quad \dot{P}_j = H_j(P) \text{ if } P_j > 0 \text{ and } H_j(P) > 0, \\ = 0 \text{ otherwise,}$$

for $j = 1, \dots, m$,

where

$$(DE.2) \quad \text{sgn } H_j(P) = \text{sgn } F_j(P) \text{ or } H_j(P) \equiv 0, \text{ the latter holding for at most one } j.$$

Since P^2 enters into F^1 , this dynamic theory is not complete. It is necessary to postulate that P^2 is determined by the expectations of the market, which in turn will be determined by past experience in some way. We shall adopt here the hypothesis of adaptive expectations (see Section 1), which can be written

$$(DE.3) \quad \dot{P}_j = a_j(P_{j-m} - P_j) \text{ for } j = m+1, \dots, n, \text{ where } a_j > 0.$$

If there is a numéraire, j_0 , then we can assume that the market has a perfect expectation of its price. This can be achieved without modifying (DE.3) by assuming that for any solution the starting values, $P_{j_0}(0)$ and $P_{j_0+m}(0)$, are the same.

Remark 1. We do not have to worry about the expected prices becoming negative. From (DE.1), $P_j(t) \geq 0$ for $j \leq m$. Then if $P_j(t) = 0$ for some $j > m$, $P_{j-m}(t) \geq 0$, so that, from (DE.3), $\dot{P}_j \geq 0$.

A point \bar{P} satisfying the conditions,

$$F^1(\bar{P}) \leq 0, \bar{P}^1 \cdot F^1(\bar{P}) = 0, \bar{P}^1 = \bar{P}^2,$$

is necessarily an equilibrium of the dynamic system (DE.1-3). We shall postulate that there exist competitive equilibria with respect to the entire set of excess demand functions (including those for future goods) for which

current and future expected prices are equal; these will be termed stationary equilibria. This assumption could be deduced from some stationarity conditions on the excess demand functions or the utility functions underlying them. For such an equilibrium, by definition,

$$F(\bar{P}) \leq 0, \bar{P}^1 = \bar{P}^2,$$

and, by (W), $\bar{P} \cdot F(\bar{P}) = 0 = \bar{P}^1 \cdot F^1(\bar{P}) + \bar{P}^2 \cdot F^2(\bar{P})$. Since each term is non-positive, $\bar{P}^1 \cdot F^1(\bar{P}) = 0$, and so, by the previous remarks, a stationary equilibrium is an equilibrium of the dynamic system (DE.1-3).

We shall in fact postulate the existence of stationary equilibria not only for the original system but also for all systems formed from it by combining a set of commodities into a composite commodity.

(SE) There exists \bar{P}^1 such that $F(\bar{P}^1, \bar{P}^1) \leq 0$; further, if we define, for a given set of commodities S and a given weight vector π_S ,

$$\begin{aligned} G_j(P_S^1, p_o^1, P_S^2, p_o^2) &= F_j(P_S^1, p_o^1 \pi_S, P_S^2, p_o^2 \pi_S) \text{ for } j \in S \\ &\text{or } j = m \in S, \\ G_o^1(P_S^1, p_o^1, P_S^2, p_o^2) &= \sum_{j \in S} \pi_j F_j(P_S^1, p_o^1 \pi_S, P_S^2, p_o^2 \pi_S), \\ G_o^2(P_S^1, p_o^1, P_S^2, p_o^2) &= \sum_{j \in \tilde{S}} \pi_j F_{j+m}(P_S^1, p_o^1 \pi_S, P_S^2, p_o^2 \pi_S), \end{aligned}$$

so that we are forming a composite commodity from a set \tilde{S} of the current commodities and another composite commodity from the corresponding set of future commodities, then there exists \bar{P}_S^1, \bar{p}_o^1 such that

$$G(\bar{P}^1, \bar{p}_o^1, \bar{P}^1, \bar{p}_o^1) \leq 0.$$

To simplify matters, we ignore (unlike in the previous parts of the paper) the difficulties of corner equilibria by postulating positive demand for free goods,

(PF) if $P_j = 0$, then $F_j(P) > 0$ ($j = 1, \dots, m$) .

Remark 2. From (PF), we cannot have $\bar{P}_j = 0$ for any j in (SE).

Hence, we certainly have a positive equilibrium, and condition (a) of Theorem 1 is fulfilled. Also, the second line of (DE.1) becomes superfluous, for suppose $P_j(t) = 0$ for some $t > 0$, and $j \leq m$. Then $F_j > 0$, while commodity j cannot be a numéraire, and therefore $H_j > 0$. Further, then $\dot{P}_j(t) > 0$, so that $P_j(t - \epsilon) < P_j(t)$ for ϵ sufficiently small. Since $P_j(t - \epsilon) \geq 0$, $P_j(t) > 0$. Hence, we can assume that $P(0) > 0$ without loss of generality.

Finally, we will assume as before that the functions $F(P)$ satisfy all the conditions (W), (H), (C), (B), and (S).

5.2. Theorem 3. Under the assumptions of Section 5.1, the dynamic system defined by (DE.1-3) is globally stable in the sense that every solution converges, provided the solutions are unique and continuous in the starting point.

Proof: As before, we use Theorem 1. Hypothesis (a) is implied by assumption (PF) (see Remark 2 of 5.1); hypothesis (c) is explicitly stated. It remains to verify hypotheses (b) and (d).

By (SE) and (PF), we can choose a positive equilibrium \bar{P} with

$$(46) \quad \bar{P}_j = \bar{P}_{j+m} \quad (j = 1, \dots, m) .$$

Suppose j is such that

$$(47) \quad P_j(t) / \bar{P}_j = \max_k P_k(t) / \bar{P}_k .$$

If $j \leq m$, we may apply Lemma 3; we must have $F_j[P(t)] \leq 0$ and therefore $H_j[P(t)] \leq 0$. If $j > m$, then, by assumption,

$$P_j(t) / \bar{P}_j \geq P_{j-m}(t) / \bar{P}_{j-m}.$$

From (46), $P_j(t) \geq P_{j-m}(t)$, and by (DE.3), $\dot{P}_j(t) \leq 0$. Since this holds for all j for which (47) holds, it follows from the proof of Lemma 4(A) that

$$(48) \quad \max_k P_k(t) / \bar{P}_k \text{ is monotonic decreasing.}$$

An exactly parallel argument applies to $\min_k P_k(t) / \bar{P}_k$, so that hypothesis (b) of Theorem 1 is verified.

We have finally to verify hypothesis (d) of Theorem 1. As in the proof of Theorem 2, we let C be the eventually constant set of components of $P(t)$ other than the numéraire. Let

$$C_1 = \{j: j \leq m, j \in C\}, C_2 = \{j: j \leq m, j+m \in C\}.$$

The set C is completely determined by C_1 and C_2 . If $j \in C_2$, then from (DE.3), $P_j(t) \equiv P_{j+m}(t)$ for t sufficiently large; since $P_{j+m}(t)$ is constant for t sufficiently large, by definition, we must have $j \in C_1$.

$$C_2 \subset C_1.$$

Let $P^*(t)$ be a limit-path of $P(t)$. We must show that at least one element not eventually constant on $P(t)$ is eventually constant on $P^*(t)$. We consider two cases. First suppose that there is at least one j in C_1 but not in C_2 . Since $P_j(t)$ is constant for t sufficiently large,

the differential equation (DE.3), with j replaced by $j+m$, becomes a single differential equation with a stable solution, so that $P_{j+m}(t)$ converges. Then by Lemma 1, $P_{j+m}^*(t)$ is constant. Since j did not belong to C_2 , this means there is at least one more eventually constant component in $P^*(t)$ than in $P(t)$.

Now suppose that $C_1 = C_2$. For t sufficiently large, $P_j(t)$ and $P_{j+m}(t)$ are constant for $j \in C_1$. From (DE.3), we must have $P_j(t) = P_{j+m}(t)$. Let the common constant values form a vector, π_{C_1} . By Lemma 1, we must also have

$$P_j^*(t) \equiv P_{j+m}^*(t) \equiv \pi_j \quad (j \in C_1) .$$

Now form all the commodities in C_1 , together with the numéraire, if any, into a single composite commodity. Let V be the remaining current commodities. Call the new set of excess demand functions $G(P)$; they are defined as in the statement of assumption (SE). The corresponding dynamic system is

$$(49) \quad dP_j/dt = H_j(P) = H_j(P_{C_1}^1, \pi_{C_1}, P_{C_1}^2, \pi_{C_1}) \quad (j \in V) ,$$

$$dP_{j+m}/dt = a_{j+m}(P_j - P_{j+m}) \quad \text{for } j \in V .$$

The system (49) is satisfied by $P(t)$, for t sufficiently large, and $P^*(t)$. It also satisfies all the conditions of the Theorem, in view of (SE), so that we can conclude that (48) holds for this system. Let W be the variables of (49) (including the composite numéraire). Then, by Lemma 1,

$$(50) \quad \max_{k \in W} P_k^*(t) / \bar{P}_k \equiv \bar{\mu} , \min_{j \in W} P_j^*(t) / \bar{P} \equiv \underline{\mu} .$$

In the next subsection (5.3), we will demonstrate (Lemma 5) that from (50), we can infer the existence of $j', j'' \in W$ such that

$$(51) \quad P_{j'}^*(t) / \bar{P}_{j'} \equiv \bar{\mu}, P_{j''}^*(t) / \bar{P}_{j''} \equiv \underline{\mu},$$

If $\bar{\mu} = \underline{\mu}$, then $P_j^*(t)$ must be constant for all $j \in W$ and therefore for $j \in \tilde{C}_1$ or j such that $j - m \in \tilde{C}_1$. Since the other variables are certainly constant, then all variables are constant and (d) is satisfied.

If $\bar{\mu} > \underline{\mu}$, then j' and j'' are distinct and cannot both be the composite numéraire. Neither j' nor j'' belong to C so that (d) is again verified.

5.3. We have only to prove the following lemma:

Lemma 5. If \max_j (resp., \min_j) $P_j(t) / \bar{P}_j$ is identically constant, then there is some j' such that

$$P_{j'}(t) / \bar{P}_{j'} \equiv \max_j P_j(t) / \bar{P}_j \text{ for } t \text{ sufficiently large.}$$

Proof: Suppose not.

$$(52) \quad \max_j P_j(t) / \bar{P}_j \equiv \bar{\mu}.$$

Then in particular,

$$(53) \quad P_j(t) < \bar{\mu} \bar{P}_j \text{ in some time interval, for } j \leq m.$$

From (DE.3),

$$(54) \quad P_{j+m}(t) = e^{-a_j t} [P_{j+m}(0) + a_j \int_0^t e^{a_j s} P_j(s) ds].$$

Since, by (52),

$$P_{j+m}(0) \leq \bar{\mu} \bar{P}_j, P_j(s) \leq \bar{\mu} \bar{P}_j \text{ for all } s,$$

it follows from (54) that

$$P_{j+m}(t) \leq e^{-a_j t} [\bar{\mu} \bar{P}_j + a_j \bar{\mu} \bar{P}_j \int_0^t e^{a_j s} ds] = \bar{\mu} \bar{P}_j .$$

(Indeed, $P_{j+m}(t)$ is in effect a weighted average of $P_{j+m}(0)$ and past values of $P_j(s)$.) Further, the strict inequality holds if $P_j(s) < \bar{\mu} \bar{P}_j$ over some s -interval. In view of (53), we can conclude

$$(55) \quad P_{j+m}(t) / \bar{P}_j < \bar{\mu} \text{ for all } t \text{ sufficiently large.}$$

Let

$$(56) \quad M(t) = \{j: P_j(t) / \bar{P}_j = \bar{\mu}\}, \quad F = \{j: j > m\} .$$

From (56), (55) can be written

$$(57) \quad M(t) \text{ is disjoint from } F \text{ for } t \geq t_0 ,$$

for suitable t_0 .

Among values of $t \geq t_0$, choose t_1 so that the number of elements of $M(t)$ is minimal. Let

$$(58) \quad M = M(t_1) , \quad N = \{j: j \leq m , j \notin M\} .$$

Suppose $\bar{V}[P(t), \bar{P}: M] > \bar{V}[P(t), \bar{P}: N]$. In view of (57), it follows that $M(t) \subsetneq M$. But for $t \geq t_0$, it follows from the choice of t_1 that $M(t)$ cannot be a proper subset of M .

$$(59) \quad \text{If } \bar{V}[P(t), \bar{P}: M] > \bar{V}[P(t), \bar{P}: N] , \text{ then for } t \geq t_0 , M(t) = M .$$

Suppose for some $t \geq t_1$,

$$\bar{V}[P(t), \bar{P}: M] = \bar{V}[P(t), \bar{P}: N] .$$

Let t_2 be the smallest such t . Then, since $\bar{V}[P(t_1), \bar{P}: M] > \bar{V}[P(t_1), \bar{P}: N]$ by definition of t_1 , we must have $t_2 > t_1$, and

$$(60) \quad \bar{V}[P(t), \bar{P}: M] > \bar{V}[P(t), \bar{P}: N] \text{ for } t_1 \leq t < t_2,$$

$$(61) \quad \bar{V}[P(t_2), \bar{P}: M] = \bar{V}[P(t_2), \bar{P}: N].$$

Let

$$(62) \quad N(t) = \{j: j \in N, P_j(t) / \bar{P}_j = \bar{V}[P(t), \bar{P}: N]\}.$$

From (59) and (60), $M(t) = M$ for $t_1 \leq t < t_2$. In a sufficiently small left-hand neighborhood of t_2 , it follows from (61) and (55) that $P_j(t) / \bar{P}_j$ has a greater value for $j \in N(t)$ than for $j > m$. Hence, if the commodities (present and future together) are ranked in increasing order of $P_j(t) / \bar{P}_j$, the elements of M will be the highest, those in $N(t)$ the next highest.

At the same time, if M contained the numéraire, the conclusion of the lemma would hold, contrary to supposition. Since $P_j(t) / \bar{P}_j \equiv \bar{\mu}$ for $j \in M$ for $t_1 \leq t < t_2$, $P_j(t)$ is identically constant over this interval, so that $H_j[P(t)] \equiv 0$. Since j is not the numéraire,

$$(63) \quad F_j[P(t)] \equiv 0 \text{ for } t_1 \leq t < t_2.$$

In view of the ranking found in the preceding paragraph, it follows from (63) and Lemma 3 that $F_j[P(t)] \leq 0$ for all $j \in N(t)$. This implies that $H_j[P(t)] \leq 0$ for all $j \in N(t)$, and hence, by the reasoning used in the proof of Lemma 4(A),

$\bar{V}[P(t), \bar{P}: N]$ is monotone decreasing in a left-hand neighborhood of t_2 .

But then it is impossible, as implied by (60) and (61) that

$$\bar{V}[P(t), \bar{P}: N] < \bar{\mu} \text{ for } t < t_2, \bar{V}[P(t_2), \bar{P}: N] = \bar{\mu}.$$

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